Technical Report

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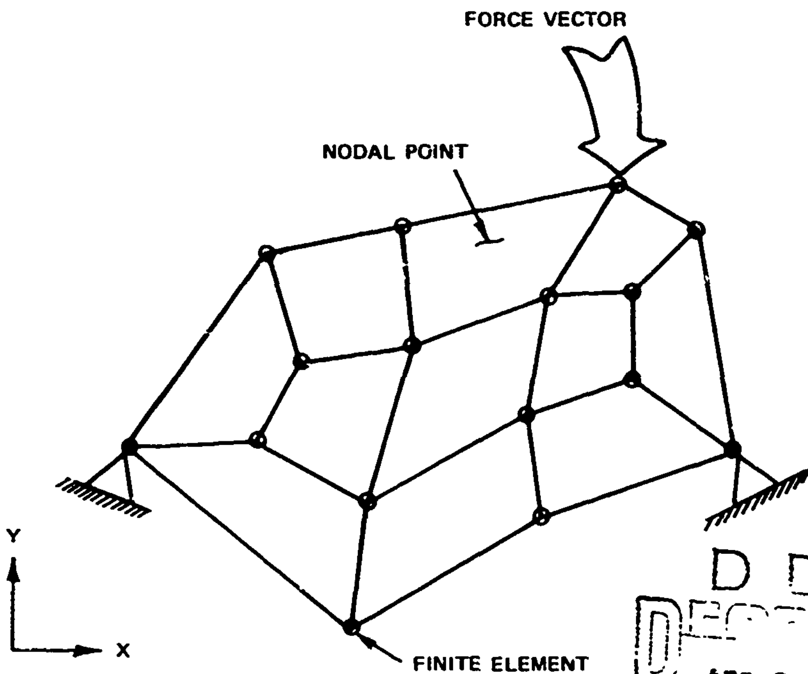

PROBABILISTIC MODEL FOR MATERIAL STRENGTH VARIATION AND SIZE EFFECT

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PROBABILISTIC MODEL FOR MATERIAL STRENGTH VARIATION AND SIZE EFFECT

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by

Salah Nosseir and Masanobu Shinozuka

ABSTRACT

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INTRODUCTION

Measures of stiffness and strength of most structural materials are known to exhibit considerable spatial statistical variation. The inherent difference in the behavior of two test coupons extracted from the same steel bar is but a typical example. These differences are more conspicuous in composite materials such as structural concrete and modern fiber-reinforced composites. The random nature of the resistance together with uncertainties involved in the estimation of the loading conditions made it necessary for the structural engineer to adapt the so-called "specified minimum strength" that could lead to an uneconomical design. The specified minimum strength depends only on the material regardless of the size of the structural member to be designed. Accordingly, the variation of the overall resistance with the size of a structural member cannot be predicted.

The weakest link hypothesis was introduced to explain the influence of the size of a structural member on the magnitude and dispersion of strength.[1,2] Classically, this hypothesis is associated with "links" having statistically independent strengths; this overly simplifies reality and limits the scope of its application. The development of an appropriate criterion for the variation of resistance with size for brittle materials would not only be beneficial in structural modeling, but would also provide the means for a better interpretation of the scatter observed in test results and, thereby, lead to better design.

The overall objective of this study is to establish a methodology to deal with uncertainties arising from the randomness of the properties of structural materials with particular emphasis on structural concrete. To this end, a new and more general interpretation of the statistical strength distribution is proposed; this interpretation includes the classical interpretation as a special case. To illustrate the applicability of this approach to composite materials, a probabilistic model was constructed for the digital simulation of concrete strength applicable even in the range of nonlinear inelastic response. This model is compatible with the finite element analysis and, therefore, is useful in immediate applications such as in planning experiments, in parametric studies involving pertinent strength variables, in structural modeling, and in analysis and design of structural systems.

PROBLEM FORMULATION

Statistical Size Effect

Classical Approach. For simplicity, let the effect of statistical anisotropy be ignored and consider the statistical scatter of a ***measure of strength***, denoted by X_o, of a hypothetical reference volume element, dv, of a microscopic magnitude. The quantity X_o may be interpreted as the strength at a point within the material.

According to the weakest link hypothesis, if X_n denotes the strength of an aggregate of a material consisting of n such microscopic volume elements, then X_n has to be the minimum among the strengths of these volume elements constituting the aggregate considered. Under the assumption that these strengths are ***identically*** and ***independently*** distributed with the distribution function $F_{X_o}(a)$, the distribution function $F_{X_n}(a)$ of X_n is

$$F_{x_n}(a) = 1 - [1 - F_{x_o}(a)]^n \tag{1}$$

where, by definition, the distribution function $F_x(a)$ of the random variable X is equal to the probability that X will be less than or equal to a.

As n increases, n dv = v becomes finite, and X_v (written for X_n) represents the strength of a piece of material of a finite volume, v. It is usually postulated that the distribution function of X_v approaches the Weibull distribution, which is one of the asymptotic distribution functions of smallest values, of the form

$$F_{x_v}(a) = 1 - \exp\left[-v\left(\frac{a}{A}\right)^k\right] \tag{2}$$

where A and K are positive constants depending only upon the material properties.

The analytical form of Equation 2 is such that the distribution function of the strength associated with volume v_1 is located to the left of that associated with volume v_2 if $v_1 > v_2$ as schematically shown in Figure 1. This indicates the statistical size effect: ***an aggregate of a smaller volume exhibits, on the average, a larger strength than that of a larger volume, as long as the two aggregates consist of the same material.***

The validity of such a postulate depends, among other things, on the mutual independence of the strengths of reference volume elements.

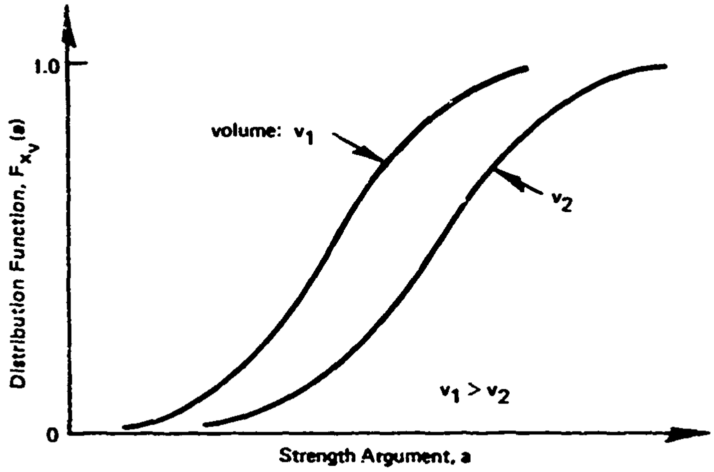


Figure 1. Effect of size on strength distribution function.

Proposed Approach. This section describes the proposed interpretation of the statistical strength distribution and size effect; this interpretation includes the classical approach as a special case. The essence of the proposed approach lies in the interpretation that the strength is a multidimensional random process.

For simplicity, consider first the case of a one-dimensional strength variation, that is, the variation of strength along a line arbitrarily chosen within the material. Without loss of generality, let the x-axis represent this line and $X_o(x)$ denote the strength of the material at a point for which the coordinate is x. The proposed approach regards $X_o(x)$ as a one-dimensional random process. One may visualize $X_o(n)$ as the axial strength variation of a thin rod in which the radial strength variation is negligible.

On the basis of experimental evidence, it appears reasonable to assume that $X_o(x)$ is a homogeneous Gaussian process with a mean value X_m and a deviation $X(x)$ from X_m; hence, $X_o(x)$ can be expressed as follows:

$$X_o(x) = X_m + X(x) \tag{3}$$

Therefore, the standard deviation, denoted by σ_o, of $X_o(x)$ equals that of $X(x)$.

Consider the length L of the thin rod under consideration to be much larger than a reference length, d. Suppose that the spectral density function $\sigma_o^2 g(k)$ and the autocorrelation function $\sigma_o^2 r(\xi)$ of the process $X(x)$ take a normalized form such as that shown in Figure 2; that is:

$$
\left.
\begin{aligned}
g(k) &= \frac{1}{k_m} - \frac{|k|}{k_m^2} & |k| < k_m \\
g(k) &= 0 & |k| > k_m \\
r(\xi) &= \left[\frac{\sin\left(\frac{\lambda_m \xi}{2}\right)}{\frac{\lambda_m \xi}{2}} \right]^2 &
\end{aligned}
\right\} \quad (4)
$$

where k_m [illegible] imum wave number, ξ is the distance between two points along [illegible] od, and λ_m ($= 2\pi/k_m$) is the wavelength associated with k_m. The [illegible] ength λ_m can serve as a correlation distance or a distance along the x-axis over which appreciable autocorrelation occurs.

According to the proposed approach, the ratio between the correlation d[illegible]ce λ_m and the reference length **d** plays a major role in the interpretation of the statistical strength variation and size effect. Figure 3 depicts a sample of the strength variation $X_o(x)$ with $X(x)$ having a correlation distance considerably shorter than the reference length ($\lambda_m/d<<1$), whereas another sample of $X_o(x)$ with $X(x)$ possessing a correlation distance considerably longer than **d** ($\lambda_m/d>>1$) is schematically shown in Figure 4. As a limiting case, a situation can be considered where the correlation distance approaches zero and the autocorrelation takes the form $\sigma_o^2\,\delta(\xi)$, where $\delta(\xi)$ is the Dirac delta function. This corresponds to the classical approach of independent strength variation. Another limiting case produces a complete coherence in which the correlation distance is infinite. Consequently, no strength variation is observed along the x-axis within one specimen; however, the strength still may vary from one specimen to another.

Aside from the two limiting cases mentioned above, the proposed approach permits correlation among the strengths of various elements. The trend of such strength correlation is, in general, at least similar to that of the autocorrelation function of $X(x)$. In the thin rod example, if the correlation distance λ_m is much shorter than the reference length **d**, no significant correlation is expected between even the neighboring elements; whereas, if λ_m is much longer than **d**, significant correlation is expected over a number of successive elements whose total length is approximately equal to λ_m. Accordingly, for the purpose of simulating the strength variation, the selection of the parameters defining the process $X_o(x)$ may be guided by the fact that the spectral density and the autocorrelation functions form a Fourier transform pair. Hence, the spectral density is flatter or wide-banded (more concentrated or narrow-banded) if the correlation distance is shorter (longer).

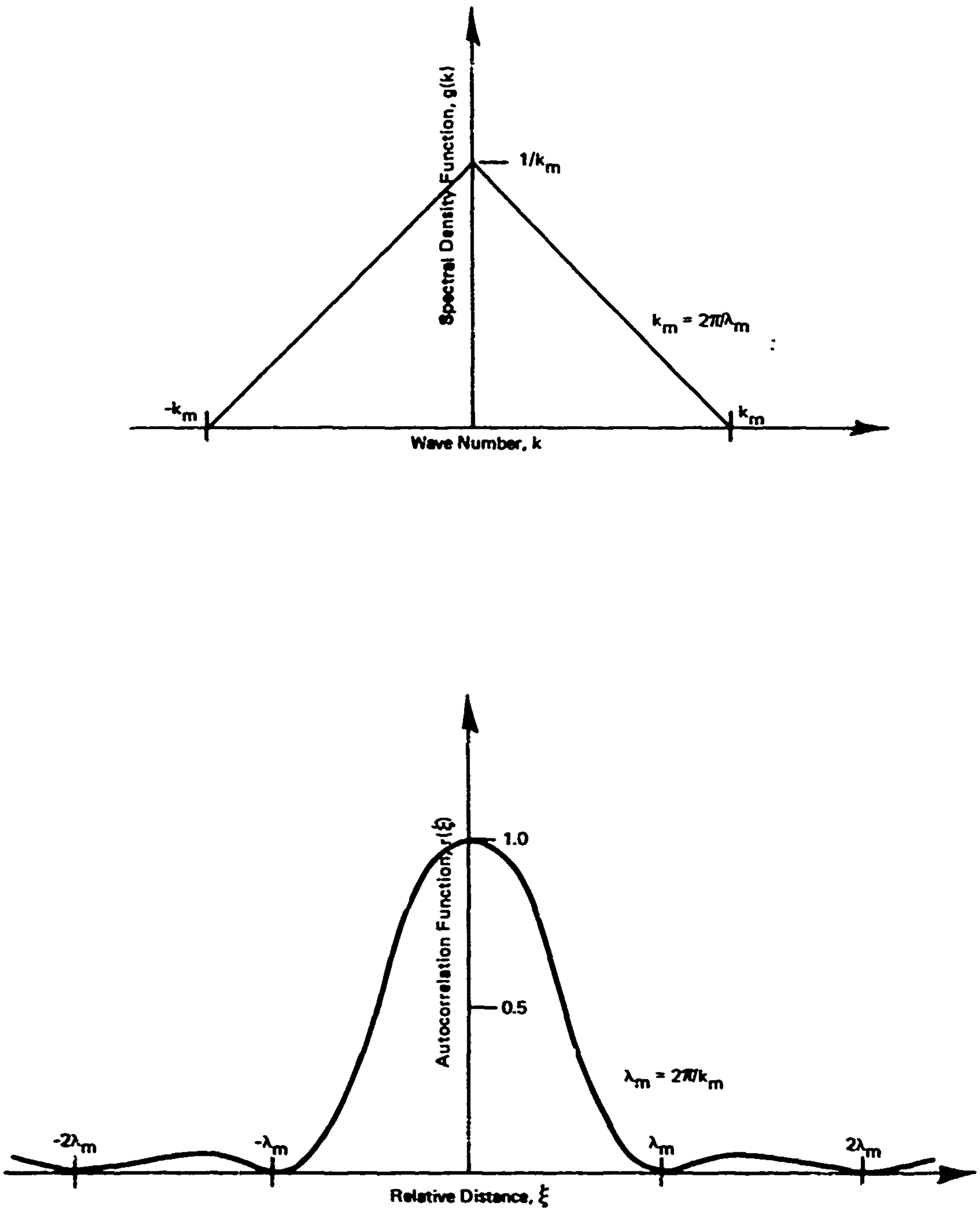


Figure 2. Graphs of spectral density and autocorrelation functions

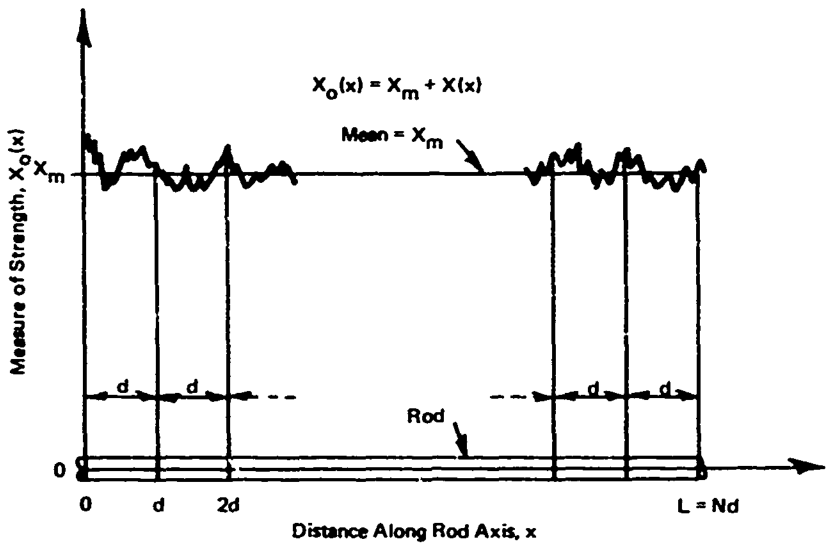


Figure 3. Strength variation for small λ_m/d ratio.

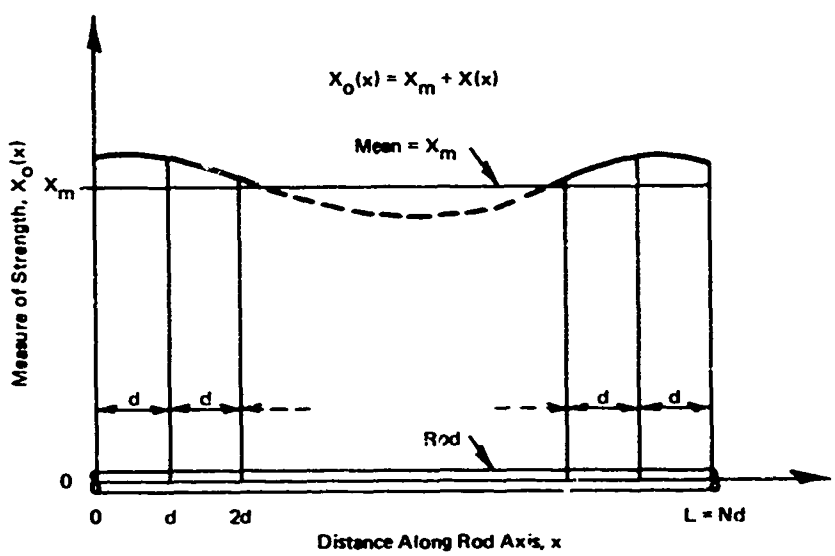


Figure 4. Strength variation for large λ_m/d ratio.

To interpret the statistical size effect, consider again the example of a thin rod for which the strength variation in the radial direction is negligible when compared with that in the axial direction. Invoking the weakest link hypothesis together with the proposed interpretation of the strength, the strength of a rod-element of a length **d** is then the minimum value of $X_o(x)$ or X_m plus the minimum value of $X(x)$ in that element. Therefore, the strength of a larger element is likely to be smaller when the size effect is taken into account.

The preceding idea of the statistical size effect can easily be expanded into more realistic two- or three-dimensional situations. Consider, for example, the simple beam shown in Figure 5 where the thickness c is small enough that the statistical variation of the strength across the thickness is negligible. If the strength at any point (x, y) is treated as a function $X_o(x, y)$, then, according to the weakest link hypothesis, the strength of an element of the beam, such as **e** of area $A_e = dd'$ (Figure 5), is the minimum value of $X_o(x, y)$ within the area A_e.

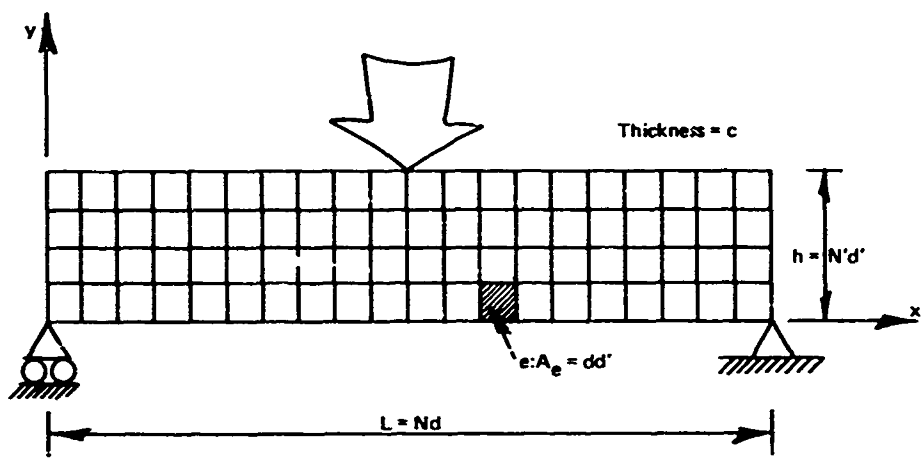


Figure 5. A simple beam:grid for strength variation.

It again appears reasonable to assume that $X_o(x, y)$ is a homogeneous (although two-dimensional this time) Gaussian process consisting of the mean value X_m and the deviation $X(x, y)$ from X_m:

$$X_o(x, y) = X_m + X(x, y) \tag{5}$$

where $X(x, y)$ is a homogeneous, two-dimensional Gaussian process with mean zero, standard deviation σ_o, and "normalized" autocorrelation function $r(\xi_1, \xi_2)$. The variables ξ_1 and ξ_2 indicate the separation between two points (x_1, y_1) and (x_2, y_2): $\xi_1 = x_2 - x_1$ and $\xi_2 = y_2 - y_1$. Since $X(x, y)$ is homogeneous, the double Fourier transform $g(k_1, k_2)$ of $r(\xi_1, \xi_2)$ of the form

$$g(k_1, k_2) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r(\xi_1, \xi_2)\, e^{-i(k_1 \xi_1 + k_2 \xi_2)} d\xi_1\, d\xi_2 \qquad (6)$$

is real and nonnegative by virtue of Bochner's theorem.[3]

The expansion of the same idea into the three-dimensional case involving $X_o(x, y, z)$ is straightforward.

Digital Simulation of Strength Variation

The strength variation, which is interpreted herein as a random process, is simulated utilizing the method developed in Reference 4.

Consider first the one-dimensional strength variation $X_o(x) = X_m + X(x)$. Since X_m is a constant, $X(x)$ with mean zero and autocorrelation $\sigma_o^2\, r(\xi)$ is the process to be digitally simulated.

According to Reference 4, if $G(x)$ denotes a homogeneous Gaussian process with mean zero, unit standard deviation, and autocorrelation $r(\xi)$, then $G(x)$ can be expressed as

$$G(x) = \left(\frac{2}{M}\right)^{1/2} \sum_{i=1}^{M} \cos(k_i x + \Phi_i) \qquad (7)$$

where k_i = random variables identically and independently distributed with density function $g(k)$, which is the Fourier transform of $r(\xi)$

M = a large positive integer (say, 100)

Φ_i = random variables identically and independently distributed with uniform density between zero and 2π and is independent of k_j $(j = 1, 2, \ldots M)$

Equation 7 can be utilized for the digital simulation of $G(x)$ through the replacement of the random variables k_i and Φ_i by their realizations. Since $X(x)$ can be written as

$$X(x) = \sigma_o G(x) \tag{8}$$

It is evident that a realization of the minimum value of $X(x)$ in an interval of length d is equal to that of $G(x)$, in the same interval, multiplied by σ_o.

A similar technique can be applied to simulate the strength variation over a two-dimensional domain. In this case, the strength is interpreted as the process $X_o(x,y)$ which can be decomposed into the mean value X_m and the deviation $X(x)$ as given by Equation 5. Following the same line of reasoning as in the one-dimensional case, a two-dimensional homogeneous Gaussian process $G(x,y)$ can be simulated digitally in the form

$$G(x,y) = \left(\frac{2}{M}\right)^{1/2} \sum_{i=1}^{M} \cos(k_{1i}x + k_{2i}y + \Phi_i) \tag{9}$$

where k_{1i} and k_{2i} = jointly distributed random variables with identical joint density function $g(k_1, k_2)$ irrespective of i; independent of k_{1j} and k_{2j} $(i \neq j)$

M = a large positive integer

Φ_i = random variable uniformly distributed between zero and 2π; independent of Φ_j $(i \neq j)$ and of any set of k_{1j} and k_{2j}

Replacing k_{1i}, k_{2i}, and Φ_i in Equation 9 by their respective realizations generates a realization of $G(x,y)$. A realization of $X(x,y)$ can then be determined from the relation

$$X(x,y) = \sigma_o G(x,y) \tag{10}$$

Obviously, realizations of $X_o(x,y)$ can be obtained from those of $X(x,y)$ utilizing Equation 5.

Again, the expansion of the method of simulating material strength into the three-dimensional case is straightforward.

Estimation of Parameters From Observed Data

This section deals with the procedure for establishing the link between the results from the digital simulation and the data observed in laboratory experiments. Such a link would provide the parameters for defining the random process. This is done by fitting the distribution function resulting from the assumed random process to that constructed from the observed data.

Let $F_d(a)$ denote the distribution function of the minimum value of the process $X_o(x)$ in an element of length **d** (Figures 3 or 4). Then, $F_d(a)$ depends in general not only on **d** but also on the parameters defining the process $X_o(x)$. When $X_o(x)$ is a homogeneous Gaussian process with mean X_m, standard deviation σ_o, and autocorrelation function $\sigma_o^2\, r(\xi)$ as assumed in this study, the distribution function $F_d(a)$ depends on X_m, σ_o, and the analytical form of $r(\xi)$ as well as **d**. The procedure for establishing such a dependence is described below.

For the thin rod example, select an autocorrelation function $r(\xi)$ for an interval of length **L** of the rod consisting of **N** subintervals **d**. In this case, **L** = **N d** is chosen in such a manner that **N** is a reasonably large positive integer (say, 200). Simulate the corresponding homogeneous Gaussian process **G(x)** as outlined in the previous section.[4] The minimum value of the realization of **G(x)** in each of the subintervals **d** constitutes an element of a sample of size **N**. Because **N** is large, the distribution function, denoted by $F_d^o(a)$, of the minimum value of **G(x)** over the interval **L** can be inferred from such a sample with good engineering confidence.

As stated earlier, the strength variation **X(x)** can be simulated as $\sigma_o\, G(x)$, and that a realization of the minimum value of **X(x)** in an interval of length **d** is equal to that of **G(x)** in the same interval multiplied by σ_o. This indicates that the corresponding distribution function of the minimum for $X_o(x) = X_m + X(x)$ can simply be obtained from the distribution function $F_d^o(a)$ of the minimum for **G(x)** as follows:

$$F_d(a) \;=\; F_d^o\left(\frac{a - X_m}{\sigma_o}\right) \tag{11}$$

Obviously, a realization of **G(x)** generated over the interval **L** can be utilized to estimate the distribution function $F_d(a)$ with different values of **d** as long as $L/d >> 1$. Accordingly, in applying this Monte Carlo method for the estimation of $F_d(a)$, only one realization of **G(x)** of sufficient length has to be generated for a class of processes with the normalized autocorrelation function identical to that of **G(x)**.

One should recall that the distribution function $F_d(a)$ given by Equation 11 was derived from theoretical consideration rather than experimental observations, and that herein $F_d(a)$ depends on a tentative set of parameters [$r(\xi)$, X_m, σ_o, and **d**]. The selection of the appropriate values for the elements of a set calls for a laboratory experiment such as the one outlined below.

Consider, again, the thin rod example where the strength is measured for **N** subintervals **d** of an interval of length **L**. This time, however, the experiment is conducted in the laboratory for an "actual" thin rod. This experiment

provides a sample of N strength values taken from the population of strength values $\bar{F}_d(a)$. From the results of such an experiment, it is possible to estimate approximately the analytical form of $r(\xi)$. This makes it possible to find $F_d^o(a)$ by the method of digital simulation described earlier. The remaining two parameters (the mean X_m and the standard deviation σ_o) can then be estimated in such a way that

$$F_d^o\left(\frac{a - X_m}{\sigma_o}\right) \approx \bar{F}_d(a) \qquad (12)$$

As far as the problem in hand is concerned, the validity of Equation 12 insures conformity of concept to reality.

The preceding approach can readily be extended into a two-dimensional case. Recalling the example of the simple beam, Figure 5, the distribution function $F_A(a)$ of the resisting strength of the element e of area $A_e = d\,d'$ can be obtained from the distribution function $F_A^o(a)$:

$$F_A(a) = F_A^o\left(\frac{a - X_m}{\sigma_o}\right) \qquad (13)$$

where $F_A^o(a)$ is the distribution function of the minimum value of $G(x, y)$ within the element e of dimensions $d\,d'\,c$. The distribution function $F_A^o(a)$ is to be inferred on the basis of digital simulation of $G(x, y)$ over a two-dimensional region (for example, the rectangular area $L\,h$ of Figure 5) consisting of a large number of subregions (elements $d\,d'$, Figure 5). The autocorrelation function $r(\xi_1, \xi_2)$ of $G(x, y)$ is required for such a simulation. Similar to the one-dimensional case, the parameters $r(\xi_1, \xi_1)$, X_m, and σ_o can be found from observed results of strength measurements performed, for example, on each of N elements of size $d\,d'\,c$ of a rectangular region of thickness c.

Again, the expansion of this procedure into the three-dimensional case is straightforward.

With an additional effort of performing simple transformations, one can treat without much difficulty the strength process which is not Gaussian. However, in the face of the limited amount of information available at the present time, such an effort may not be worth the expense.

Finite-Element Failure Analysis

This section demonstrates that it is possible to perform a more realistic structural failure analysis when the proposed statistical strength model is utilized.

When a failure analysis is sought, a failure criterion is always required. Failure criteria normally call for comparison between a computed or measured load-dependent action (for example, stress, strain, . . . etc.) and a known or assumed ultimate value of such action. This ultimate value is usually referred to as strength or resistance. In what follows, it is assumed that (1) the load-dependent action is computed by means of the finite element method,[5] and (2) the resistance is determinable from a process such as $X_o(x, y)$ for which the parameters X_m, σ_o, and $r(\xi_1, \xi_2)$ are estimated (based on strength measurements) with reasonable engineering accuracy. Without loss of generality, it is also assumed that the failure criterion is applicable for actions within the finite elements rather than at the *nodal points* of the idealized structure. The ensuing discussion addresses itself mainly to the question of determining the resistance of each of the finite elements.

Consider a two-dimensional failure analysis of an arbitrary structure, Figure 6, for which the strength variation is negligible in the direction normal to the plane of interest. For the purpose of discussion, a coarse grid of quadrilateral finite elements is shown in Figure 6; the selection of the "best" finite element grid is considered outside the scope of this study.

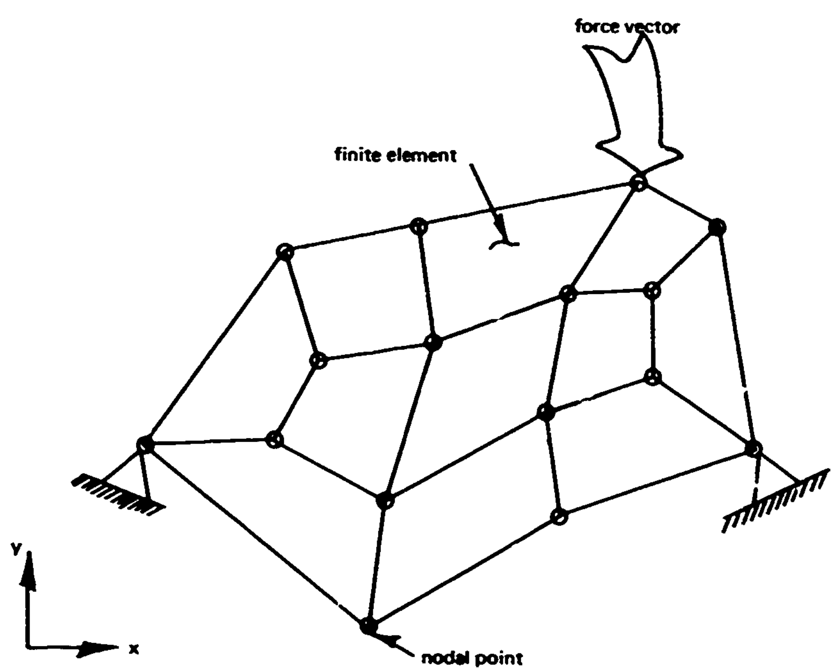


Figure 6. Finite element idealization of a two-dimensional structure.

Utilizing the approach outlined earlier, a number of realizations of $X_0(x, y)$ can be generated digitally. Each realization will be equivalent to a structure chosen arbitrarily from the population of structures with the specified form of statistical strength variation. Obviously, the value of $X_0(x, y)$ can be evaluated for any pair of finite values of x and y. However, when x and y are restricted to the region representing the structure, the corresponding values of $X_0(x, y)$ become of special significance for they represent the spatial strength variation of the structure. For such digital simulation of strength, an arbitrary grid may be selected, and the value of $X_0(x, y)$ may be determined at the ***intersection points*** of the grid. In what follows, the grid for digitally evaluating values of $X_0(x, y)$ will be referred to as the ***strength grid***.

For the special case in which the strength grid coincides with that constituting the finite elements, the intersection points are the nodal points of the finite element grid. Thus, the number n_e of values of $X_0(x, y)$ for points (x, y) falling within each finite element is equal to the number of exterior nodal points for that element. Applying the weakest link hypothesis at the finite element level, the strength of any element is, then, the minimum of n_e values of $X_0(x, y)$ for that element. The strength of each finite element thus obtained can be compared with the load-dependent action obtained from the finite element solution to determine whether failure at the element level has occurred.

In the preceding discussion, the strength grid was assumed to coincide with the finite element grid. However, this is not necessary because the grid of finite elements depends mainly on the expected stress gradient within the structure under consideration: ***steeper gradients call for finer finite element grids***. Similarly, the selection of a strength grid should depend on the manner in which the strength varies. More precisely, the size of the strength grid should be in harmony with the autocorrelation function: ***shorter correlation distances should call for finer strength grids***. For expediency, however, it is not advisable to make the strength grid coarser than the finite element grid regardless of the length of the correlation distance.* The reason is that difficulty will arise in defining the strength of any finite element for which n_e is zero.

In summary, it is recommended that each finite element contain a strength grid such that the nodal points of that element also will be the intersection points of that grid. The size of the strength grid within each element should be made finer as the correlation distance gets shorter. In this context, finer grid means that the number of the intersection points of the strength grid exceeds the number of nodal points of the finite element as shown in Figure 7.

* The only exception is the limiting case where the correlation distance is infinitely long; in this case, all the elements have the same strength, and, therefore, the construction of a strength grid is irrelevant.

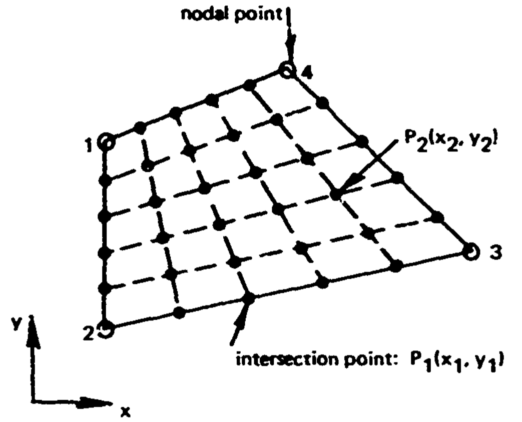


Figure 7. A strength grid for a quadrilateral finite element.

As a typical example, consider a failure analysis of a two-dimensional concrete structure. The strength of an element of plain concrete of a certain size under a two-dimensional state of stress can be expressed in terms of a failure envelope, Figure 8, in the $\sigma_1 - \sigma_2$ plane of principal stresses.[6] The domain in the $\sigma_1 - \sigma_2$ plane enclosed by such an envelope represents the various states of stresses that can be imposed without causing a failure of that element. The equation of this envelope can be written in the form

$$R(x, y; \theta) = X_o(x, y)\, R_o(\theta) \qquad (14)$$

where θ is a measure of the stress path, $R_o(\theta)$ represents a reference strength, and $R(x, y; \theta)$ is a random process describing the statistical variation of the strength. Equation 14 implies the hypothesis that the shape of the failure envelope is geometrically similar but its size varies statistically; this hypothesis, however, is not essential to the present analysis. According to this hypothesis, the relative positions of the failure envelopes for two intersection points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, Figure 7, with $X_o(x_1, y_1) > X_o(x_2, y_2)$ is shown schematically in Figure 8. If, for example, the reference envelope is associated with the median, the failure envelope for any arbitrary point $P(x, y)$ has a probability of 0.5 to be located outside (or inside) the reference envelope. This also implies that the probability of $X_o(x, y)$ for any point $P(x, y)$ being larger (or smaller) than unity is 0.5.

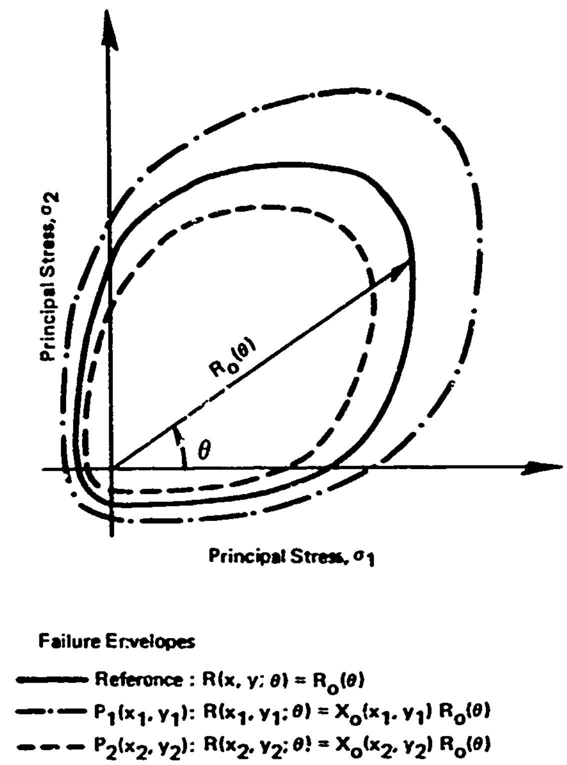


Figure 8. Envelopes for failure of concrete under biaxial stresses.

Assuming that the beam shown in Figure 5 is made of plain concrete, the present model of statistical strength variation makes it possible to simulate $X_o(x, y)$ and, therefore, to assign spatially correlated failure envelopes to the finite elements. As the central load is increased from zero, the principal stresses, from the finite element solution, are examined until "failure" of an element is identified. The value of the central load that causes this condition is recorded, and the procedure is repeated for a reasonably large number of (simulated) beams. Accordingly, one can estimate the statistical characteristics of the structural failure. Obviously, such a failure analysis is extremely useful in the prediction of structural integrity. It is important to note that the stress analysis has to be performed only once if the material properties such as the elastic constants are not statistical.

In this example of a plain concrete beam, the weakest link hypothesis was assumed to govern explicitly at the finite element level and implicitly at the global or the entire structure level. This implicit assumption at the global level may be realistic for a plain concrete beam (brittle failure) but not, in

general, for a reinforced concrete beam (ductile failure). It is to be emphasized that the proposed approach suggests the weakest link hypothesis only at the local (finite element) level but does not require such assumption at the global level.

In the preceding sections, the random process $X_0(x, y)$ was defined as a measure of the strength interpreted as the resistance associated with failure. It is equally likely, and perhaps as important, to utilize the same procedure of digital simulation when the process $X_0(x, y)$ is defined as a measure for any material property (for example, modulus of elasticity) of a finite element. In such a case, a decision has to be made as to whether the weakest link assumption is realistic even at the finite element level; an averaging procedure may provide a better alternative.

APPLICATION: DIGITAL SIMULATION OF CONCRETE MODELS

This section presents a numerical example of applying the proposed approach to investigate the effect of size on the strength variation of specimens made of concrete. Because of the lack of experimental data concerning the spatial strength variation of concrete, assumptions had to be made in order to define the strength as a random process. However, instead of being absolutely arbitrary, such assumptions were made to reflect some experimental evidence. Also discussed is the influence of the choice of the strength grid on the resulting strength of the simulation models.

Failure Level and Simulation Parameters

Failure of a structural material is normally associated with a certain level or more precisely a certain dimension at which failure is observed or estimated (for example, microscopic failure, macroscopic failure, etc.). It is general practice to consider that failure of a structural material occurs when some measure of stress (or strain) has reached or exceeded its ultimate value (usually referred to as the strength). By definition, stresses are associated with lengths or dimensions. But, strains are dimensionless; they relate displacements to some physical dimension or gage length. However, it is well known that the value of the strain is, in general, a function of the gage length. In other words, strains are dependent on dimensions in spite of their apparent nondimensionality. This paradox normally disappears when the gage length is standardized whether this is done to relate strains either to stresses or to an ultimate value for failure analysis. Accordingly, failure is associated with dimensions implicitly (or explicitly) when the failure criterion is based on strains (or stresses). The selection of

the dimension with which failure is associated is limited by the precision of the instruments available and governed by the consequences of failure (for example, collapse) at the selected dimension.

In considering structural concrete, design engineers are generally interested in avoiding major failures, that is, the failures associated with large dimensions.[7] In this case, the assumption that plain concrete behaves as a homogeneous material is reasonable. However, this assumption becomes unrealistic, and composite action needs to be accounted for when failure is associated with smaller dimensions as in the case of bond failure between cement paste and aggregate.[8,9] In considering failure at a much smaller dimension, propagation of concrete cracking is thought to initiate from capillary pores (voids in the cement paste that were originally filled by water but have not become filled by the products of hydration).[10]

Generally, as the dimension with which failure is associated decreases, the strengths and maximum stresses increase, and the maximum and minimum values occur at shorter intervals.[10] The spatial variation of a measure of one-dimensional strength of concrete as influenced by the gage length may take the form shown in Figure 9.

In order to make the numerical example of practical value to designers, the failure considered is that associated with dimensions comparable to the size of standard concrete cylinders. For such sizes, the most important parameters that influence the variation of strength with size are the maximum aggregate size and the fineness modulus.[11-13] Based on the theory of dimensional analysis, Brown proposed consideration of the maximum aggregate size in studies of size effect.[14] Obviously, many parameters such as the aggregate gradation, mix proportions, method of mixing, placing, curing, age, and rate of loading influence the strength of a concrete specimen. The question is whether such influences change with the sizes considered.

For the numerical example presented below the following assumptions were made:

1. Strength in any direction is evaluated from failure of specimens with a minimum dimension of Δ such that $\Delta = 4\,A_m$ where A_m is the maximum aggregate size.[15] Such a specimen will be referred to as the *reference specimen.*

2. The mean strength X_m, the standard deviation σ_o, and the autocorrelation function $r(\xi)$ are evaluated, in the manner discussed earlier, from specimens conforming with Assumption 1 above:

$$\Delta = d = 4A_m$$

and, therefore,

$$F_\Delta(a) = F_d(a)$$

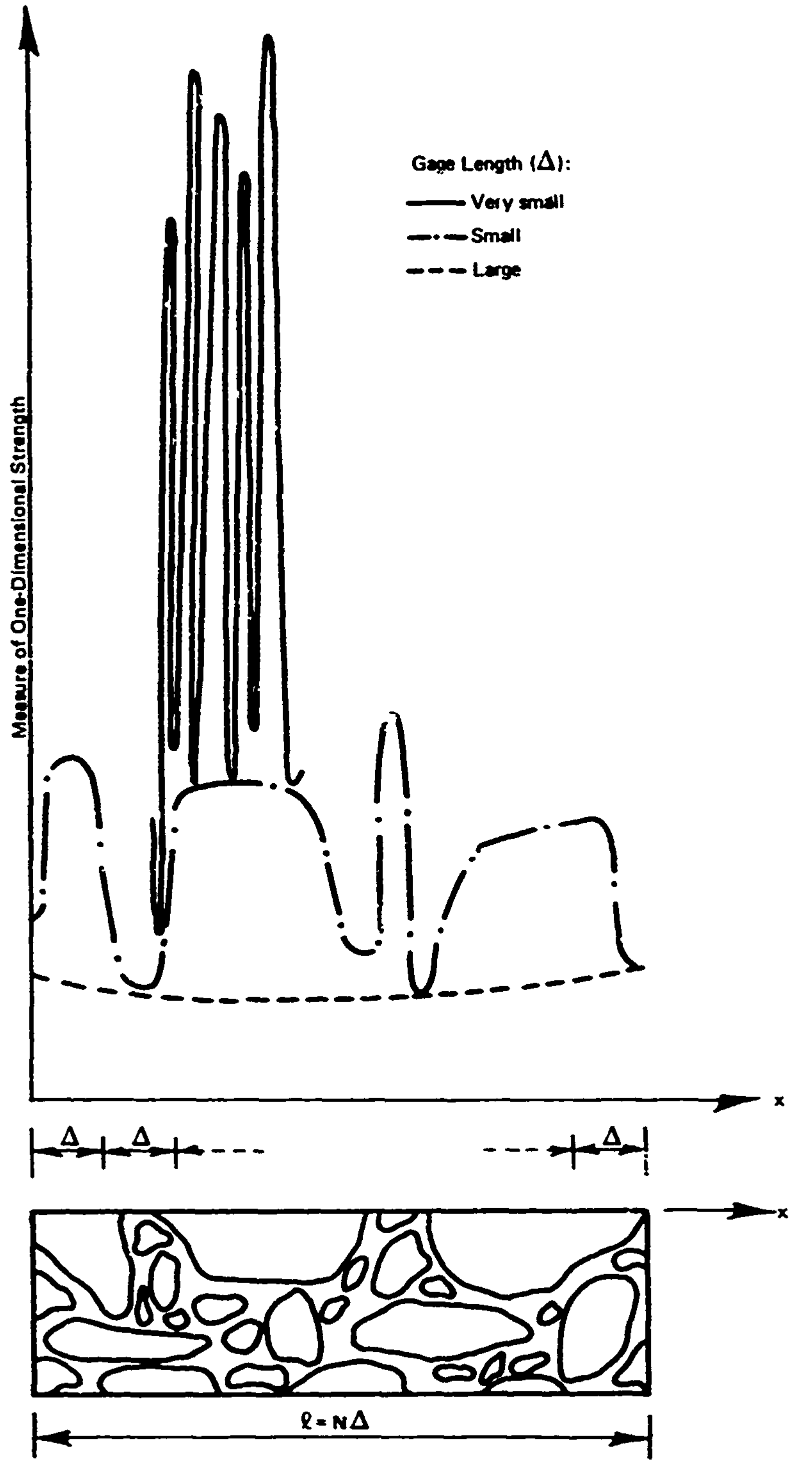


Figure 9. Influence of gage length on "observed" strength.

3. The autocorrelation function takes a form similar to that shown in Figure 2 such that

$$\lambda_m = 24 A_m$$

Description of Models

For a concrete mix with a maximum aggregate size A_m = 1/4 inch, consider a reference specimen of dimensions $\Delta\ \Delta$ c inches such that $\Delta = 4 A_m$ = 1.0 inch, and assume that the strength does not vary with he thickness c. The strength of the reference specimen is determined by applying uniform pressure along two opposite edges without constraining the movement of any edge, Figure 10. The following parameters are assumed to have been determined from testing a large sample of reference specimens:

$$X_m = 10{,}000 \text{ psi} \qquad \sigma_o = 2{,}000 \text{ psi} \qquad \lambda_m = 6 \text{ inches}$$

The proposed approach was utilized to determine the variation of strength when the specimen's dimensions, in inches, vary from 1 x 1 x c to 3 x 3 x c and to 12 x 12 x c for the same type of loading, Figure 10. For this loading condition, the stress distribution does not vary with the size of the specimen. Accordingly, the finite element grid, Figure 11a, consisted of the same number of finite elements for both 3 x 3-inch and 12 x 12-inch specimens. The size $d_e \times d_e$ of each finite element was 1 x 1 inch and 4 x 4 inches for the smaller and the larger specimens, respectively. The strength grid within each finite element, Figure 11b, was selected such that the distance d_s between the intersection points is small relative to the correlation distance λ_m. This was done so that the influence of the fineness of the strength grid on the calculated strength becomes insignificant. The strength distribution function for the 3 x 3-inch specimens was evaluated for two ratios of λ_m/d_s, namely 6 and 24. When the distribution functions for the two ratios were found to be almost

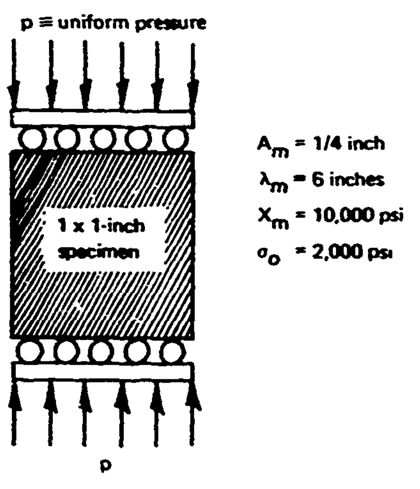
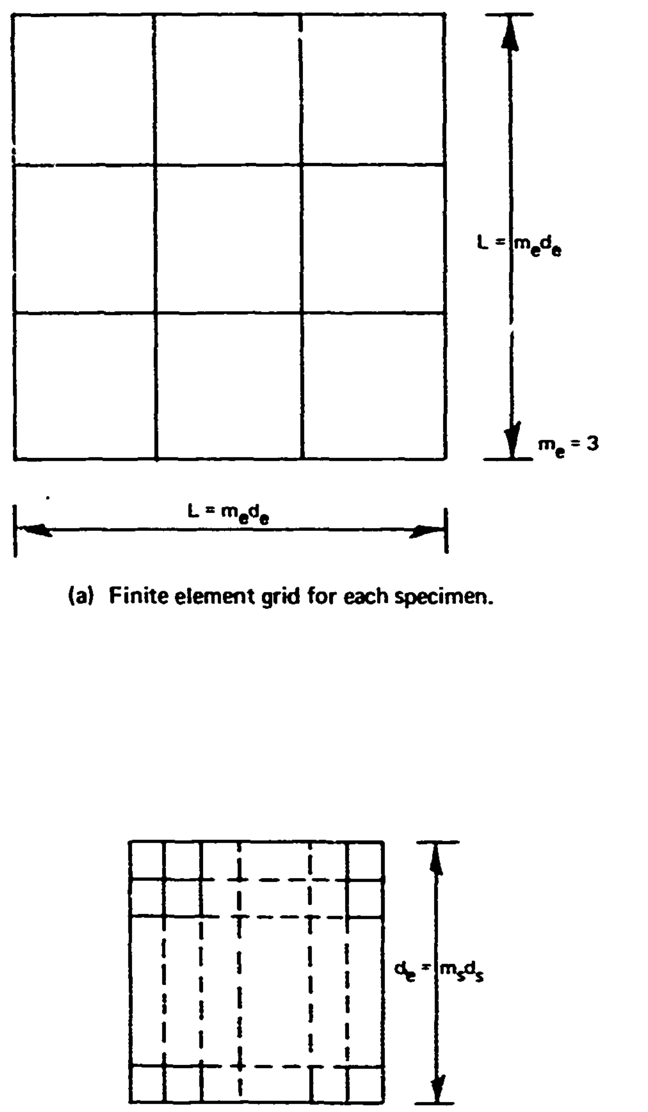


Figure 10. Reference concrete specimen.

identical, the λ_m/d_s ratio of 6 was selected for the strergth evaluation of the 12 x 12-inch specimens. Table 1 lists the properties of the specimens for which the strength distribution functions were evaluated.



(a) Finite element grid for each specimen.

(b) Strength grid for each finite element.

Figure 11. Finite element and strength grids for concrete specimens.

Table 1. Description of Specimens

Specimen Designation [L-L-(λ_m/d_s)]	L (in.)	m_e	d_e (in.)	m_s	d_s (in.)	λ_m (in.)	$\frac{\lambda_m}{d_s}$
3-3-24	3	3	1	4	0.25	6	24
3-3-6	3	3	1	1	1.00	6	6
12-12-6	12	3	4	4	1.00	6	6

The strength distribution functions were evaluated by generating 20 realizations of $X_0(x, y)$ over a 5 x 5-foot area. Each realization was equivalent to casting a 5 x 5-foot concrete slab, which could be subdivided into N specimens. The ranges of x and y provided an N of 25 for the specimens with the largest size (12 x 12 inches). Two sets of 25 specimens of 3 x 3 inches with different strength grids were then selected from the same regions of 15 x 15 inches for each realization. This was done to determine the influence of fineness of the strength grid when all other variables are held invariant.

Results and Discussion

Figure 12 shows the relations among the cumulative strength distributions determined from one of the realizations. Typically, the cumulative distributions were almost identical for the two sets of 3 x 3-inch specimens that differed only in the degree of fineness of the strength grid. For each of the 20 realizations, the distribution function for the larger specimens (12 x 12-inch) fell to the left of that of the smaller specimens. Table 2 gives the means and standard deviations obtained from each realization as well as their overall values for all 20 realizations. The distribution functions of the means are shown in Figure 13. This figure shows that the fluctuation of the means was smaller for the larger specimens. Figure 14 gives the overall strength distribution functions for the 3 x 3-inch and 12 x 12-inch specimens as determined from the simulation procedure. For reference, the idealized strength distribution function of the reference specimen is also shown in Figure 14.

These results suggest that the strength distribution function is not influenced by the fineness of the strength grid when the ratio λ_m/d_s was taken large enough. This result is very significant because the strength is an intrinsic property of the material and should not be dependent on the strength grid, which is an option in the hand of the analyst. This result could not have been obtained from the classical approach because λ_m/d_s is hypothesized to be always zero.

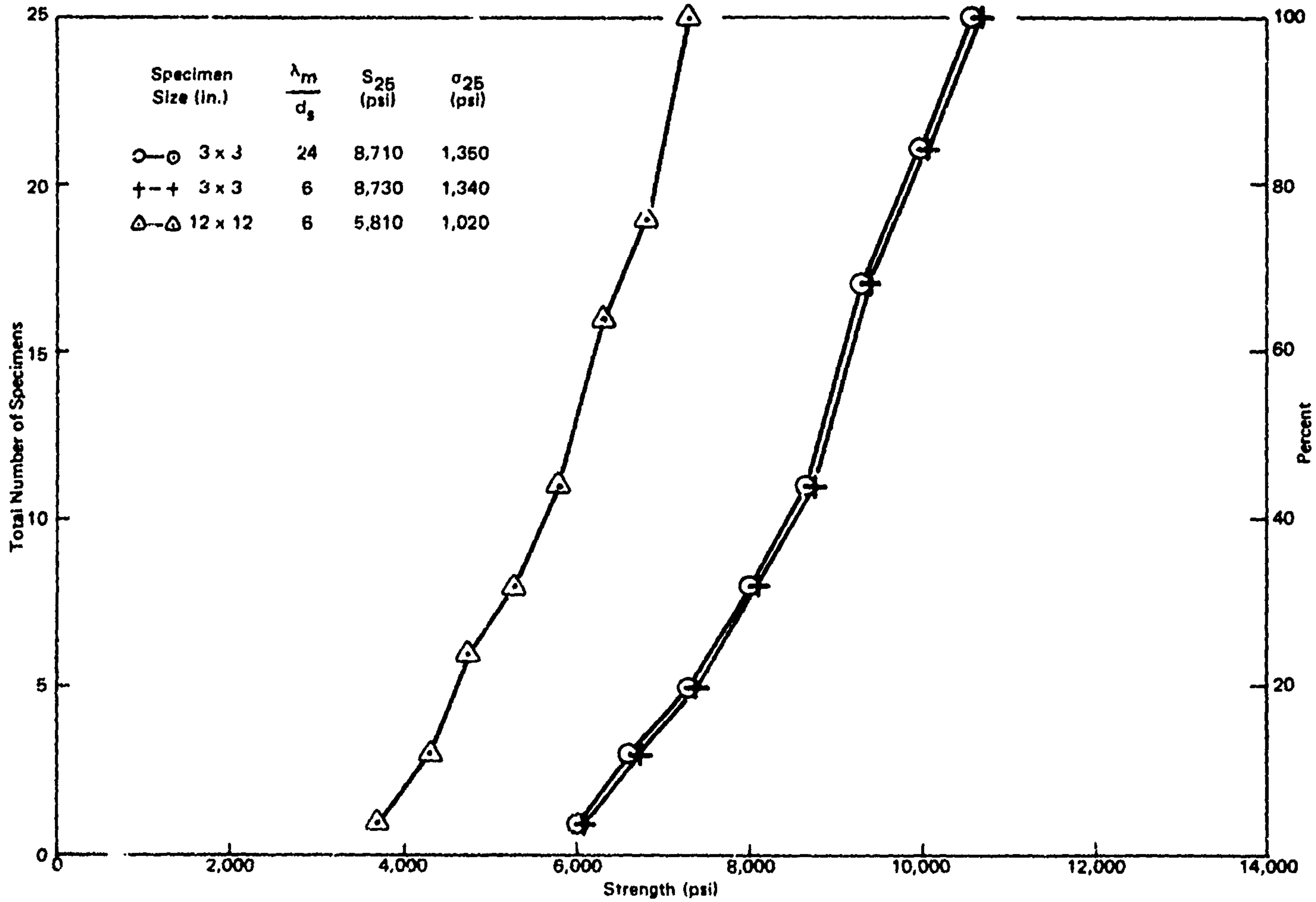


Figure 12. Cumulative strength distributions from one realization.

Table 2. Summary of Results

Realization Number	Means and Standard Deviations for—					
	3-3-24 Specimen		3-3-6 Specimen		12-12-6 Specimen	
	S_{25} (psi)	σ_{25} (psi)	S_{25} (psi)	σ_{25} (psi)	S_{25} (psi)	σ_{25} (psi)
1	8,036	1,450	8,063	1,434	5,962	1,019
2	8,616	1,118	8,637	1,107	5,973	1,096
3	8,707	1,346	8,725	1,339	5,806	1,020
4	8,169	1,394	8,182	1,382	5,678	1,338
5	8,467	1,020	8,493	1,017	6,274	1,084
6	8,414	1,172	8,424	1,171	5,877	1,409
7	8,504	1,665	8,534	1,652	6,403	[illegible]
8	8,155	1,077	8,176	1,064	6,582*	899
9	8,221	1,321	8,246	1,313	5,767	1,295
10	8,147	1,674	8,169	1,663	5,810	1,409
11	8,753	846▲	8,775	846▲	5,674	1,080
12	8,708	1,632	8,716	1,623	6,105	1,147
13	8,669	1,375	8,690	1,366	6,086	759▲
14	8,019	1,277	8,030	1,270	5,366▲	1,693*
15	9,396*	1,510	9,411*	1,497	5,675	1,533
16	7,208	1,030	7,236	1,016	6,216	902
17	7,719	1,114	7,746	1,095	6,027	1,040
18	9,006	1,301	9,038	1,287	6,094	1,243
19	8,250	1,692	8,282	1,663	6,133	1,236
20	6,536▲	2,247*	6,559▲	2,231*	5,729	1,250
	S_{500} (psi)	σ_{500} (psi)	S_{500} (psi)	σ_{500} (psi)	S_{500} (psi)	σ_{500} (psi)
all 20	8,285	1,525	8,307	1,514	5,962	1,224

* Maximum value

▲ Minimum value

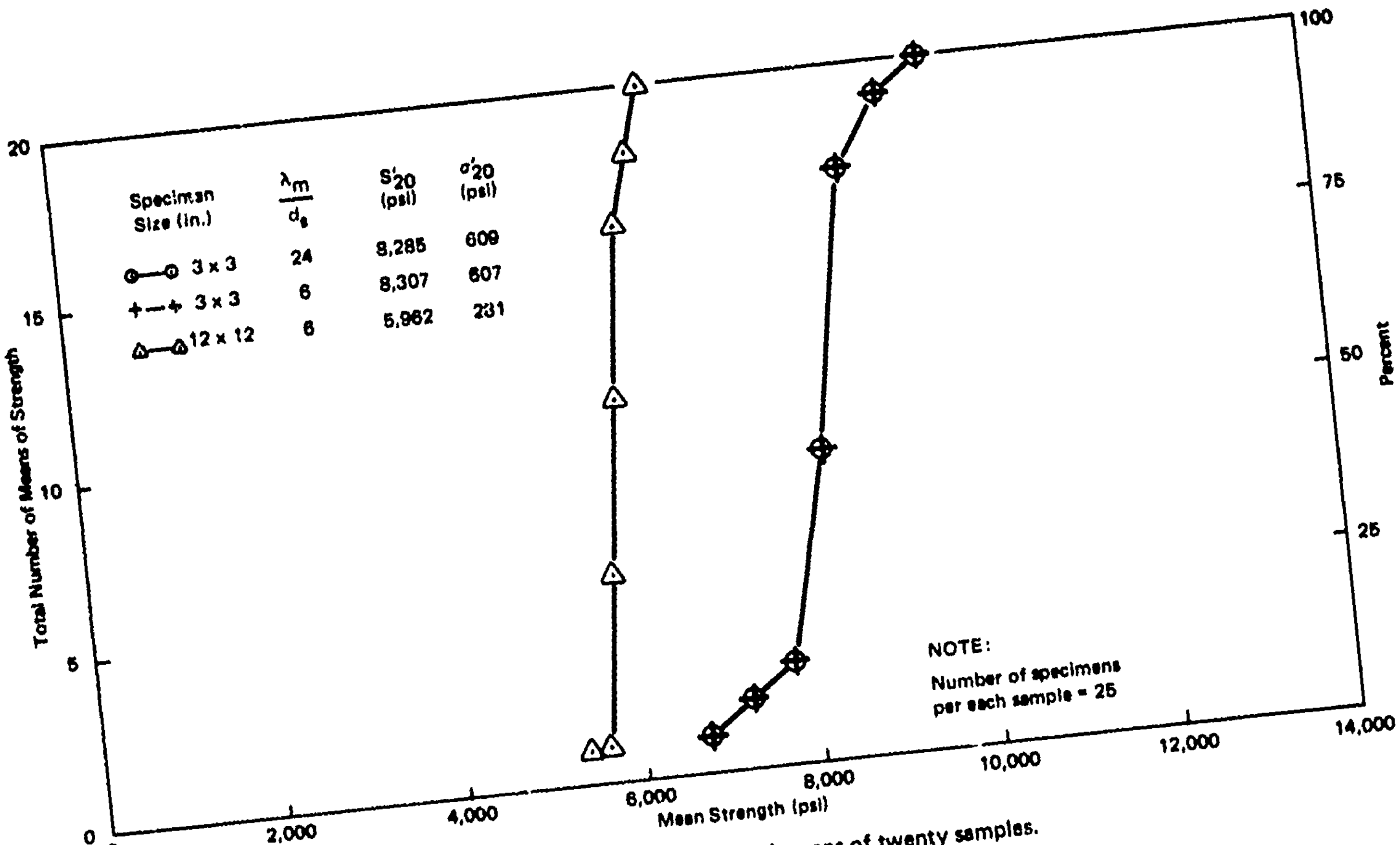


Figure 13. Cumulative distribution of means of twenty samples.

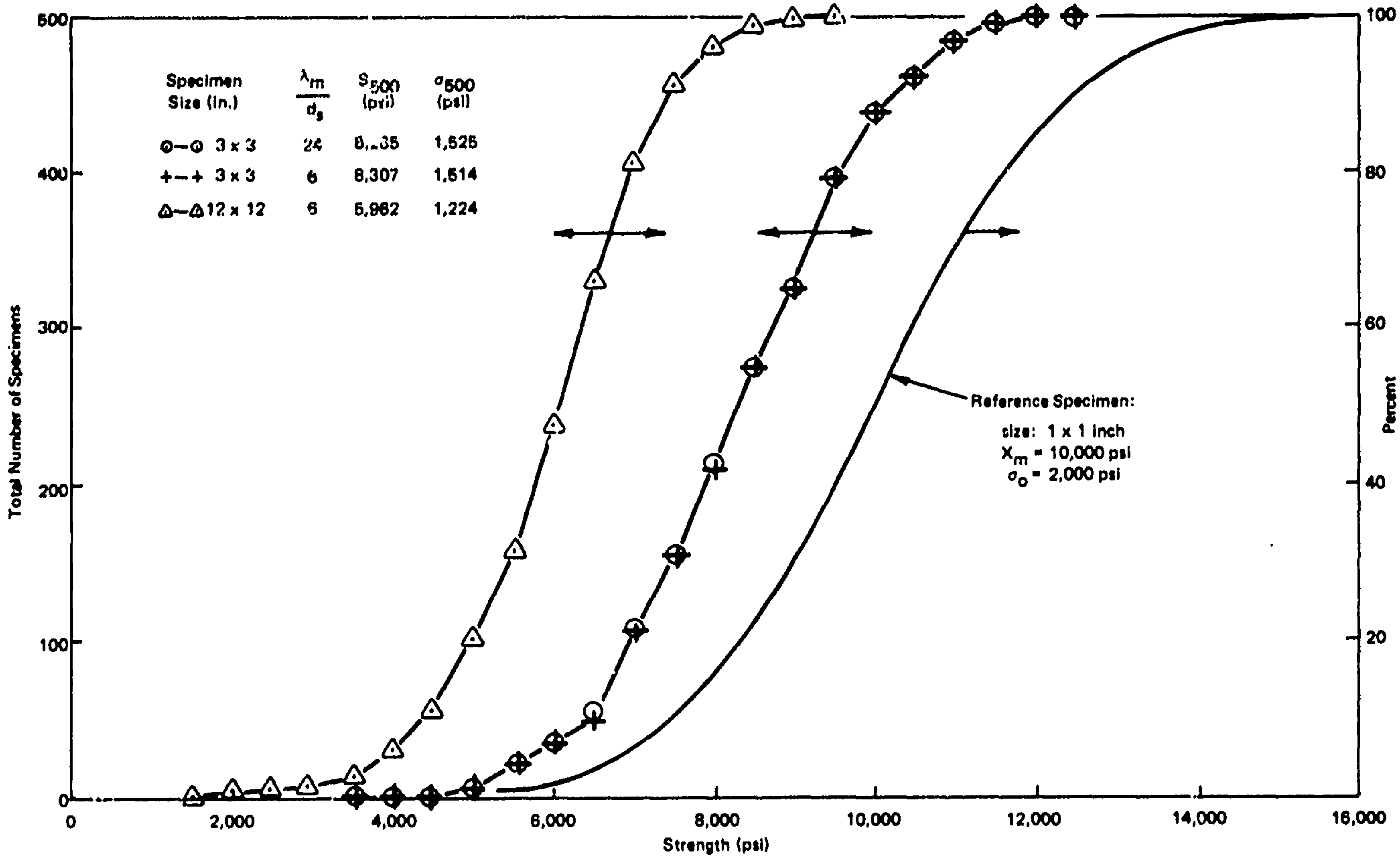


Figure 14. Cumulative strength distributions from twenty realizations.

The results presented above followed the general trend obtained from laboratory observation in that the mean value of the strength as well as the standard deviation decreases as the size of specimen increases (Figure 14). The change, in percent, of the mean strength with size (Figure 14) is in agreement with those given in Reference 15. However, one may argue that the values of the parameters chosen for the numerical example cause such agreement. This is true and, perhaps, one of the important features of the proposed approach. The ability to vary the input parameters and compare the results with those obtained experimentally may eliminate conducting new and expensive experiments to determine the spatial strength distribution in the manner previously explained.

CONCLUSION

A new interpretation of the statistical size effect is proposed based on consideration of the spatial strength variation as a random process. One limiting case of the proposed approach is the classical approach which assumes the strength to be identically and independently distributed. The other limiting case presents the engineering approximation of uniform strength. The new approach provides a powerful tool for predicting the effect of size on concrete strength in a manner that is consistent with laboratory observations.

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LIST OF SYMBOLS

A	Material constant
A_e	Area of an element
A_m	Maximum aggregate size
a	Argument of the distribution function
c	Thickness
d, d', d_e	Dimension of an element
d_s	Distance between intersection points of a strength grid
dv	An element of volume v
$F_A(a)$	Distribution function of the minimum value of the process $X_o(x, y)$ for an element of an area A_e
$F_A^o(a)$	Distribution function of the minimum value of the process $G(x, y)$ for an element of an area A_e
$F_d(a)$	Distribution function of the minimum value of the process $X_o(x)$ for an element of length d
$F_d^o(a)$	Distribution function of the minimum value of the process $G(x)$ for an element of length d
$\bar{F}_d(a)$	Distribution function $F_d(a)$ determined from laboratory experiment
$F_x(a)$	Distribution function of a random variable X
$F_{x_v}(a)$	Distribution function of a random variable x_v
F	Homogeneous Gaussian process with mean zero, unit standard deviation, and autocorrelation function r
g	Spectral density function
h	Beam height
K	Material constant
k	Wave number
k_m	Maximum wave number
L	Length of a specimen
M	Large positive integer
m	Positive integer
m_e	The ratio L/d_e
m_s	The ratio d_e/d_s
N, N'	Positive integers
n	Positive integer
n_e	Number of intersection poin[…] grid at which the value of X_o[…] ated within an element, e
r	Autocorrelation function
F	Autocorrelation function r e[…] laboratory experiment
R	A random process describing […] variation of strength
R_o	A reference strength
S_n	Mean strength of a sample of […]
S'_{20}	Mean of 20 values of S_n for […] $S'_{20} = S_{500}$[…]
V	Volume
X	A random process representi[…] of the strength from the me[…]
X_o	A random process representi[…] the strength

K	Material constant
k	Wave number
k_m	Maximum wave number
L	Length of a specimen
M	Large positive integer
m	Positive integer
m_e	The ratio L/d_e
m_s	The ratio d_e/d_s
N, N'	Positive integers
n	Positive integer
n_e	Number of intersection points of a strength grid at which the value of $X_o(x, y)$ is evaluated within an element, e
r	Autocorrelation function
$\bar{r}$	Autocorrelation function r evaluated from laboratory experiment
R	A random process describing the statistical variation of strength
R_o	A reference strength
S_n	Mean strength of a sample of n specimens
S'_{20}	Mean of 20 values of S_n for n = 25 (that is, $S'_{20} = S_{500}$)
V	Volume
X	A random process representing the deviation of the strength from the mean X_m
X_o	A random process representing a measure of the strength
X_m	The mean value of X_o
$\bar{X}_m$	Mean value of X_m determined from laboratory experiments
X_v	Random variable representing the strength of a piece of material of volume v
x, y, z	Space coordinate
Δ	Dimension of a reference specimen or gage length
δ	Dirac delta function
θ	A measure of the stress path (= arctan σ_2/σ_1)
λ_m	Wavelength associated with k_m
ξ	Distance between two material points along a coordinate axis
σ_1, σ_2	Principal stress (positive for compression)
σ_o	Standard deviation of population
σ_n	Standard deviation of specimen strength for a sample of n specimens
σ'_{20}	Standard deviation for 20 values of S_n (n = 25)